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High Reynolds number steady separated flow past a wedge of negative angle

By J. B. KLEMP† AND ANDREAS ACRIVOS
Department of Chemical Engineering, Stanford University

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According to classical boundary-layer theory, uniform flow past a semi-infinite wedge, inclined at a negative angle $\frac{1}{2}\pi\beta$ to the direction of the free stream, does not separate unless $\beta \leqslant -0{\cdot}1988$. It has been assumed, therefore, that, in the range $-0{\cdot}1988 < \beta < 0$, the flow within the boundary layer is represented by the Falkner–Skan equation, which, as was shown by Stewartson (1954), has two admissible solutions. All such solutions for $\beta < 0$ appear to be somewhat unsatisfactory, however, because they require an adverse pressure gradient, which, by becoming infinite as the corner of the wedge is approached, could lead to separation even if $\beta > -0{\cdot}1988$. In addition, the structure of the high Reynolds number flow for $\beta < -0{\cdot}1988$ has remained, to date, unresolved.

We present here a fundamentally different solution to this classical problem which eliminates the singularity in the potential region by allowing the flow to separate at the leading edge of the inclined surface. The associated flow field is then characterized by an essentially uniform free stream flowing over an inviscid and, to a high approximation, irrotational region of reverse flow in which the velocity is of $O(R^{-\frac{1}{2}})$ in magnitude, R being the Reynolds number. Mixing of these two streams is confined to a free shear boundary layer, of $O(R^{-\frac{1}{2}})$ in thickness, extending downstream from the leading edge and parallel to the direction of the undisturbed main flow. Finally, an additional boundary layer, of $O(R^{-\frac{3}{4}})$ in thickness, is shown to exist between the separated region and the surface of the wedge. Owing to the absence of a characteristic length in the problem, similar solutions to the appropriate equations describing the flow in each region are obtained and are valid for all $\beta < 0$ provided that the Reynolds number is sufficiently large. The analysis is then extended to higher order in R to increase its range of validity and to demonstrate that the proposed structure of the flow field remains self-consistent. Although the solution is developed only for a semi-infinite wedge with $\beta < 0$, it is believed that certain of its features may be of value in the analysis of other problems involving high Reynolds number separated flows.

1. Introduction

In spite of its very considerable success in dealing with laminar flow phenomena at large Reynolds numbers, laminar boundary-layer theory is at present limited, in a strict sense, to cases where separation, or 'flow breakaway', does not take

† Present address: National Center for Atmospheric Research, Boulder, Colorado.

place. In solving for the flow past a stationary solid body for example, one first obtains the solution of the Euler equations on the assumption that the surface of the body coincides with the $\Psi = 0$ streamline. The pressure profile dp/dx that is impressed on the boundary layer is then computed on the basis of this inviscid solution, and finally, the flow within the boundary layer is determined by solving the well-known laminar boundary-layer equations first derived by Prandtl. This procedure, which is described in detail in many of the standard texts on the subject, is straightforward and amenable either to an analytical approach for relatively simple body geometries or, more generally, to a numerical solution.

Fundamental difficulties arise, however, whenever the surface streamline ($\Psi = 0$) detaches itself from the surface of the body, a phenomenon often referred to as separation. As is well-known, this seems to occur whenever the pressure gradient dp/dx is even mildly positive, since, under the action of both an adverse pressure gradient and the frictional resistance of the stationary surface, it is impossible for the fluid within the boundary layer to continue along the contour of the solid wall.

From a theoretical point of view, the complications introduced by the presence of separation result primarily from the fact that the location of the $\Psi = 0$ streamline beyond its point of detachment from the surface is *a priori* unknown and must be determined through consideration of the region of reverse flow downstream of this point of breakaway. Since this streamline is displaced an order one distance from the surface in most problems of interest, fluid in the free stream then flows past an effective body whose shape may differ significantly from that of the solid object. As a result, the pressure gradient along the $\Psi = 0$ streamline will not be the same as that obtained in the absence of separation, even upstream of the point of detachment, thereby affecting the boundary-layer calculations. It is clear, therefore, that the structure of the motion within the region of reverse flow plays a crucial role, through its effect on the position of the $\Psi = 0$ streamline, in determining the flow pattern both in the inviscid region and inside the boundary layer, i.e. throughout the entire flow field.

Owing to the inherent complexity of the subject, relatively little attention has been directed to this aspect of viscous flow theory despite its importance to our understanding of a wide class of flow phenomena. As a result, to date, no self-consistent asymptotic theory, analogous to the one already available for unseparated flow, has been presented that accurately describes high Reynolds number laminar flow in which separation occurs. One of the main factors that has hindered the development of such a theory is the sparseness of relevant experimental information. Because of the instability of the motion when reverse flow occurs, experiments cannot be performed under steady-state conditions beyond a certain range of Reynolds number R, which is generally not too large, and thus little is known with any degree of certainty about the basic structure of steady separated flows at high R, especially downstream of the point of boundary-layer detachment.

We shall consider here one system which will be shown to fall into the above category. This is the flow past a wedge of negative angle $\frac{1}{2}\pi\beta$, which, assuming that separation does not occur (see figure 1), has been described, on the basis of

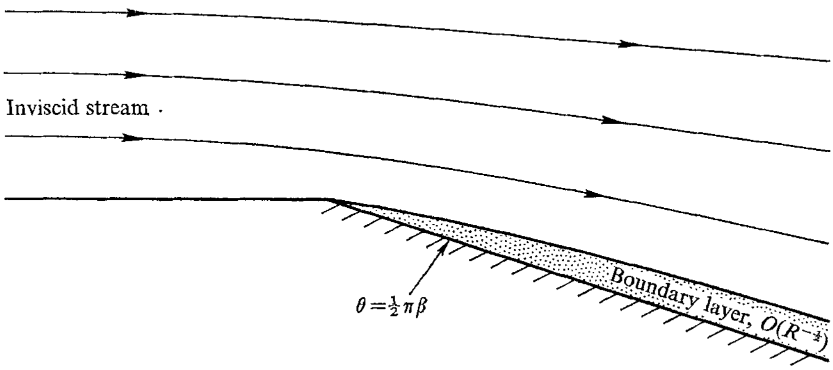

FIGURE 1. Structure of the flow for $\beta < 0$ assuming that separation does not occur.

classical boundary-layer theory, by a family of similar solutions first developed by Falkner & Skan (Rosenhead 1963). As was first shown by Hartree (1937), solutions to the appropriate laminar boundary-layer equations, or, in this case, the Falkner–Skan equation, exist without flow reversal even for $\beta < 0$, provided that $\beta > -0{\cdot}1988$. When $\beta = -0{\cdot}1988$, the skin friction is found to vanish everywhere along the surface $\theta = \frac{1}{2}\pi\beta$, whereas, when $\beta < -0{\cdot}1988$, the solutions lead to velocity profiles which indicate a reverse flow within a portion of the boundary layer. Since in the latter case, however, the velocity near the outer edge of the boundary layer exceeds that of the adjacent main stream, the solution for $\beta < -0{\cdot}1988$ cannot be accepted on physical grounds.

The Falkner–Skan equation was also investigated in some detail by Stewartson (1954), who showed that, for each β in the range $-0{\cdot}1988 < \beta < 0$, there exists an additional solution to that obtained by Hartree (1937) in which backflow occurs and for which the associated displacement thickness increases without limit as $\beta \to 0^-$. Thus, if one adopts the classical point of view according to which, as shown in figure 1, the flow consists of an inviscid main stream plus a conventional boundary layer of thickness of $O(R^{-\frac{1}{2}})$ along the surface of the wedge, one concludes that there exists a unique solution for each $\beta \geqslant 0$, two ostensibly acceptable solutions for each β in the range $-0{\cdot}1988 < \beta < 0$ and no acceptable solution for $\beta < -0{\cdot}1988$. To be sure, these conclusions result from an analysis of the similar solutions to the boundary-layer equations; however, since the physical system lacks a characteristic length, the solution to these equations, if it exists and is unique, must be of a similar form.

Upon closer examination of these results, however, it would appear that none of the similar boundary-layer solutions for $\beta < 0$ are entirely satisfactory. Specifically, since the associated velocity at the edge of the boundary layer is proportional to r^m, where r measures the distance from the corner of the wedge and $m \equiv \beta/(2-\beta)$, these solutions require, for $m < 0$, an infinite deceleration at the edge of the boundary layer (which corresponds to an infinite adverse pressure gradient) just downstream of the singular point, at $r = 0$. Hence, in view of the propensity of boundary layers to separate in the presence of even a mild deceleration of the external flow, it is not entirely clear that it is permissible to assume that the boundary layer remains attached to the wall for this particular case when the adverse gradient is initially of infinite magnitude. In fact, one might equally well suppose that the boundary layer would become detached at the origin, $r = 0$, for all $\beta < 0$, thereby removing the singularity from the potential-flow solution. This, we recall, would be in line with other instances of high Reynolds number flow, for example, flow past an airfoil with a sharp trailing edge, where it appears that the effect of viscosity acting within the boundary layer is to establish precisely that value of the circulation which either removes or at least renders milder the singularity that would otherwise exist in the solution for the main stream (Batchelor 1967, p. 437).

At any rate, even if one were prepared to accept the family of Falkner–Skan solutions for $\beta > -0{\cdot}1988$, it is evident that there exists a need for extending the analysis to the case $\beta < -0{\cdot}1988$, whose flow structure has, up to now, remained unresolved.

It is the purpose of the present paper, then, to present a fundamentally different solution to this classical problem which, in contrast to that obtained from conventional boundary-layer theory, applies for all $\beta < 0$ and for which the external potential flow is devoid of singularities. As sketched in figure 2, the flow field will be found to consist of four distinct regions: I, a conventional irrotational region of flow past an effective body of thickness of $O(R^{-\frac{1}{2}})$; II, a boundary layer separating the main stream from an essentially stagnant fluid; III, an inviscid and, to a high approximation, irrotational region of reverse flow occupying the space $\frac{1}{2}\pi\beta < \theta < 0$ in which the velocity is of $O(R^{-\frac{1}{2}})$ in magnitude; IV, a boundary layer of $O(R^{-\frac{1}{4}})$ in thickness along the surface of the wedge, $\theta = \frac{1}{2}\pi\beta$.

Although, strictly speaking, the solution to be developed will apply only to the particular case of flow past a wedge with $\beta < 0$, it is believed that certain of its features may be sufficiently general to enter into the analysis of other problems involving steady reverse flows at high Reynolds numbers.

2. Basic structure of the flow

We consider here the uniform flow of an incompressible fluid along a flat plate whose surface suddently becomes inclined at an angle $\frac{1}{2}\pi\beta$ away from the free stream. In order to analyse the same physical situation as that described by the Falkner–Skan solution for negative wedge angles, the fluid is assumed to slip freely along the surface of the plate upstream of the point of inclination. (Since this particular boundary condition is of rather limited physical interest, more realistic conditions will be discussed in § 4.) The variables are non-dimensionalized through the appropriate use of the uniform free-stream velocity U and a length scale l, which may be chosen arbitrarily since no characteristic length appears in the problem. In this manner, the Reynolds number R is defined by $R = Ul/\nu$, where ν is the kinematic viscosity.

For the reasons mentioned in the introduction, our analysis begins with the assertion that separation occurs at the leading edge ($r = 0$) of the inclined surface. In turn, since the resulting solution should evidently be independent of

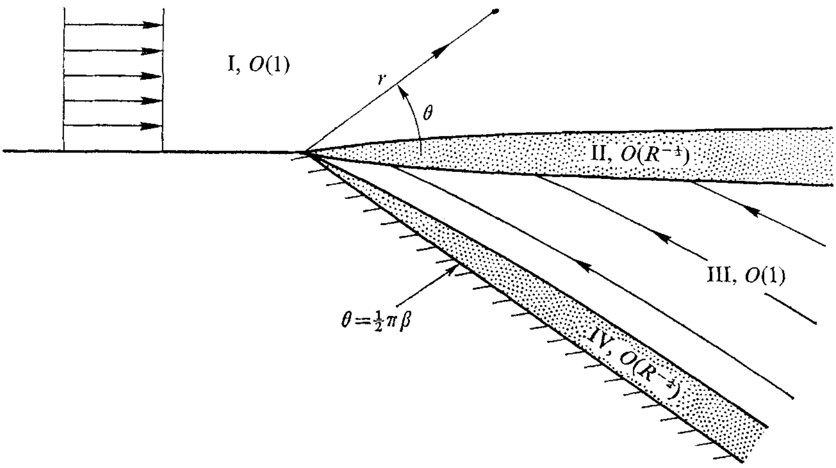

FIGURE 2. Structure of the flow for $\beta < 0$ according to the present solution (the thickness of each region being as indicated).

the choice of the characteristic length l, we require that, at large R, the $\Psi = 0$ streamline dividing the irrotational main stream from the separated region should lie along a straight line originating at $r = 0$. In fact, it is clear upon further reflexion that this streamline must be located along the $\theta = 0$ axis, i.e. parallel to the direction of the uniform free stream. Otherwise, according to potential theory, the main-stream velocity would become infinite either far downstream or at the leading edge, depending on whether the angle of inclination of the $\Psi = 0$ streamline was positive or negative.

This important conclusion regarding the location of the $\Psi = 0$ streamline allows us next to deduce the basic structure of the flow which is shown in figure 2. To a first approximation, the free stream I remains undisturbed downstream of separation and since there is no pressure gradient along the $\Psi = 0$ streamline, fluid in the separated region III remains at rest. These two regions are separated by a boundary layer II of $O(R^{-\frac{1}{2}})$ in thickness which is located along the $\theta = 0$ axis. This flow structure is then similar to that of a uniform jet discharging into a quiescent stream. Of course, region III is not truly stagnant; fluid being entrained into the boundary layer from below must come from downstream, hence a reverse flow is generated in III with velocity of $O(R^{-\frac{1}{2}})$. However, since the flow in III is inviscid, it will not be possible to satisfy the no-slip condition at $\theta = \frac{1}{2}\pi\beta$, and therefore an additional boundary layer IV must appear along the inclined surface.

From this brief description, it is already apparent that the solutions in the various regions will be strongly interdependent; hence the analysis must proceed in a definite order. With this in mind, we begin with the free shear boundary layer II. Since, to first order, the velocity in I is uniform and that in III is negligibly small, the solution in II corresponds to that presented by Lessen (1949) and by Lock (1951). Thus the stream function is given by

$$\Psi_{\mathrm{II}}(r,\theta) = (r/R)^{\frac{1}{2}} f_2(\eta), \quad \eta = (Rr)^{\frac{1}{2}}(\theta - \theta_s), \tag{2.1}$$

where $f_2(\eta)$ satisfies the familiar Blasius equation with the boundary conditions $f_2(0) = f_2'(-\infty) = 0$ and $f_2'(\infty) = 1$. The cylindrical polar co-ordinates (r, θ) are as indicated in figure 2, and $\theta_s(r; R)$ represents the position of the $\Psi = 0$ streamline. As will be seen, $\theta_s(r; R)$ is of $O(R^{-\frac{1}{2}})$ to a first approximation; moreover, since it must be independent of the choice of the characteristic length l,

$$\theta_s(r; R) = \alpha_1 (Rr)^{-\frac{1}{2}}, \tag{2.2}$$

where α_1 is a constant to be determined.

From Lock's (1951) solution we have that

$$\Psi_{\mathrm{II}}(r, \eta) \xrightarrow[\eta \to -\infty]{} -\gamma_1 (r/R)^{\frac{1}{2}}, \quad \gamma_1 = 1{\cdot}2385. \tag{2.3}$$

Consequently, an $O(R^{-\frac{1}{2}})$ flow is induced in the inviscid separated region III, which being, in general, rotational can be determined from the solution of

$$\nabla^2 \Psi_{\mathrm{III}} = -\omega(\Psi_{\mathrm{III}}), \quad \Psi_{\mathrm{III}}(r, 0) = -\gamma_1 (r/R)^{\frac{1}{2}}, \quad \Psi_{\mathrm{III}}(r, \tfrac{1}{2}\pi\beta) = 0, \tag{2.4}$$

with ω denoting the vorticity in III, which is a function only of Ψ_{III}. In view of the absence of a natural characteristic length, we seek a solution of the form

$$\Psi_{\mathrm{III}}(r, \theta) = (r/R)^{\frac{1}{2}} f_3(\theta), \tag{2.5}$$

where the r dependence is dictated by the boundary condition in (2.4). Moreover, (2.5) is consistent with (2.4) only if ω is of the form $A\Psi_{\mathrm{III}}^{-3}$, but since this would imply that ω becomes infinite as $\theta \to \frac{1}{2}\pi\beta$, we suppose, subject to *a posteriori* verification, that $A = 0$. To this order then, the separated region is irrotational. With this simplification, the solution to (2.4) can readily be obtained, and yields for $f_3(\theta)$

$$f_3(\theta) = \gamma_1 \frac{\sin \frac{1}{2}(\theta - \frac{1}{2}\pi\beta)}{\sin \frac{1}{4}\pi\beta}. \tag{2.6}$$

The velocity in the radial direction is then

$$u_r = \frac{1}{r}\frac{\partial \Psi_{\mathrm{III}}}{\partial \theta} = \frac{\gamma_1}{2}(Rr)^{-\frac{1}{2}} \frac{\cos \frac{1}{2}(\theta - \frac{1}{2}\pi\beta)}{\sin \frac{1}{4}\pi\beta} < 0, \tag{2.7}$$

which clearly verifies the presence of backflow in the separated region. To illustrate the structure of this flow, we have plotted in figure 3 the streamlines in III for $\beta = -\frac{1}{2}$. Of course, the entrainment requirements of the free shear boundary layer are independent of β, and thus, as β is increased towards zero, the velocity of the reverse flow will correspondingly increase in magnitude. In fact, u_r becomes proportional to β^{-1} as $\beta \to 0$, thereby limiting the range of validity of the present analysis when β becomes small. We shall discuss this point in more detail after completing our derivation of the first-order solutions in each region.

Since, according to (2.7), the radial velocity in III does not vanish as $\theta \to \frac{1}{2}\pi\beta$, it is apparent that a viscous boundary layer IV in which the radial velocity is of $O[(Rr)^{-\frac{1}{2}}]$ must develop along the inclined surface at $\theta = \frac{1}{2}\pi\beta$. Applying the

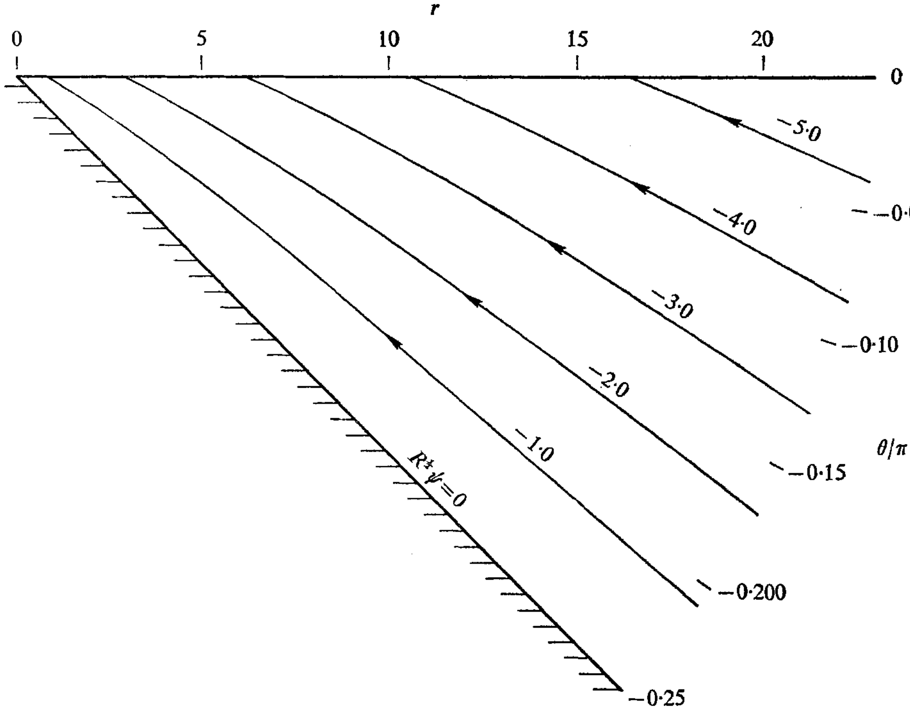

FIGURE 3. Streamline pattern in separated region III for $\beta = -\frac{1}{2}$.

usual techniques of boundary-layer analysis (Rosenhead 1963), we find that the proper similar form for Ψ_{IV} is given by

$$\Psi_{IV}(r,\zeta) = \frac{r^{\frac{1}{4}}}{R^{\frac{3}{4}}}\left[-\frac{\gamma_1}{2}\operatorname{cosec}\frac{\pi}{4}\beta\right]^{\frac{1}{2}} f_4(\zeta), \quad \zeta = (Rr)^{\frac{1}{4}}\left[-\frac{\gamma_1}{2}\operatorname{cosec}\frac{\pi}{4}\beta\right]^{\frac{1}{2}}\left(\theta - \frac{\pi\beta}{2}\right), \tag{2.8}$$

where $f_4(\zeta)$ satisfies

$$\left.\begin{aligned} &4f_4''' + f_4 f_4'' - 2[1 - f_4'^2] = 0 \\ \text{with} \quad &f_4(0) = f_4'(0) = 0, \quad f_4'(\infty) = -1, \end{aligned}\right\} \tag{2.9}$$

and thus is independent of β. This last boundary condition arises from the requirement that the radial velocity in IV as $\zeta \to \infty$ must match with the limit of (2.7) as $\theta \to \frac{1}{2}\pi\beta$. Since $f_4'(\infty) < 0$, the $O[(Rr)^{-\frac{1}{2}}]$ flow in IV accelerates toward the leading edge while, at the same time, the boundary layer becomes thinner. As is evident from (2.8), this viscous layer is of $O(r^{\frac{3}{4}}R^{-\frac{1}{4}})$ in thickness for $\beta = O(1)$.

Next, in considering the form of (2.9) as $\zeta \to \infty$, we can easily show that, in the limit,

$$f_4(\zeta) \xrightarrow[\zeta\to\infty]{} -\zeta + \lambda_1 + O(\zeta^{-3}), \tag{2.10}$$

where, from the integration of (2.9), $\lambda_1 = 1{\cdot}2585$. It is interesting to note that, as indicated by (2.10), $f_4''(\zeta)$ decays like ζ^{-5} for large ζ, and that, consequently, the vorticity in the domain of overlap between III and IV is given by

$$\omega = \frac{1}{r^2}\frac{\partial^2\Psi_{IV}}{\partial\theta^2} \xrightarrow[\zeta\to\infty]{} R^{-\frac{3}{2}}r^{-\frac{5}{2}}\left(\theta - \frac{\pi\beta}{2}\right)^{-5} \xrightarrow[\theta\to\frac{1}{2}\pi\beta]{} R^{-4}\Psi_{III}^{-5}.$$

Since any vorticity in III must be matched to the adjacent boundary layers, we would thus expect that the first contribution to the vorticity in III will be of $O(R^{-4}\Psi_{III}^{-5})$, thereby justifying our earlier assertion that, to the indicated order of approximation, ω in (2.4) is indeed negligible for $R \gg 1$. To determine the origin of this vorticity, we consider the flow in III for a large but fixed Reynolds number. By choosing a typical streamline in the interior of III at some order-one distance from the leading edge and following it downstream against the direction of flow, we find that its distance from the surface of the inclined plate increases in proportion to $r^{\frac{1}{2}}$ as $r \to \infty$. On the other hand, since the thickness of IV is of $O(R^{-\frac{1}{4}}r^{\frac{3}{4}})$, it is clear that the streamline in question will emerge from region IV when r is of $O(R)$ in magnitude. Thus, the flow in III enters the region from the boundary layer below at an $O(R)$ distance downstream and with a certain amount of vorticity, which, however, is very small.

To complete our discussion of the basic features of the flow, we shall now examine the first correction to the uniform flow in region I. This will also involve consideration of the position of the $\Psi = 0$ streamline. We begin by noting that the value of the stream function at $\theta = 0$ is obtained through matching with the boundary-layer solution in II, which, in the limit as $\eta \to \infty$, is given by

$$\Psi_{II}(r,\eta) \underset{\eta\to\infty}{\longrightarrow} (r/R)^{\frac{1}{2}}(\eta-\gamma_2) = r(\theta-\theta_s)-\gamma_2(r/R)^{\frac{1}{2}}, \tag{2.11}$$

where, from Lock's (1951) solution, $\gamma_2 = 0{\cdot}5289$ and θ_s is defined by (2.2). Thus, letting $\Psi_I(r,\theta) = r\sin\theta + \Psi_I^{(1)}(r,\theta)$, we have that

$$\nabla^2\Psi_I^{(1)} = 0, \quad \Psi_I^{(1)}(r,0) = -(\alpha_1+\gamma_2)(r/R)^{\frac{1}{2}},$$
$$\Psi_I^{(1)}(r,\pi) = 0, \quad r^{-1}\Psi_I^{(1)}(r,\theta)\to 0 \quad \text{as} \quad r\to\infty,$$

which, when solved for $\Psi_I^{(1)}(r,\theta)$, yields

$$\Psi_I(r,\theta) = r\sin\theta + (r/R)^{\frac{1}{2}}f_1(\theta), \quad f_1(\theta) = -(\alpha_1+\gamma_2)\sin\tfrac{1}{2}(\pi-\theta). \tag{2.12}$$

In seeking to determine α_1, which is related to the position of the $\Psi = 0$ streamline, we consider the pressure balance across the free shear boundary layer. Since both the pressure in III and the pressure drop across II are of $O[(Rr)^{-1}]$, the $O[(Rr)^{-\frac{1}{2}}]$ pressure term in I must vanish as $\theta \to 0$, thereby requiring that $\partial\Psi_I^{(1)}/\partial\theta = 0$ at $r = 0$. But clearly, from (2.12), this condition is satisfied regardless of the value of α_1, and consequently, this constant cannot be obtained from the first-order pressure balance across II. In dealing with this question, Ting (1959) proposed a method for calculating α_1 based on a higher order analysis of the boundary-layer equations, but unfortunately, as shown by Klemp & Acrivos (1972*b*), this procedure will not, in fact, allow α_1 to be determined uniquely since, in the higher order analysis, higher order terms must also be added to (2.2). For this reason, if the free stream is truly semi-infinite, α_1 remains indeterminate. On the other hand, if a boundary is imposed on the free stream at any arbitrarily large distance from II, then the $O(R^{-\frac{1}{2}})$ term in $\theta_s(r;R)$ can be calculated, although it may no longer have the form of (2.2). Furthermore, it has been shown by Klemp (1971) that, when a free-slip wall is placed parallel to the uniform flow at an unspecified distance from the free shear boundary layer, the requirement that $\partial\Psi_I^{(1)}/\partial\theta = 0$ at $r = 0$ leads to an integral equation for θ_s whose solution is given by (2.2) with $\alpha_1 = -\gamma_2$. This corresponds to $\Psi_I^{(1)}(r,\theta) \equiv 0$, and also means that the vertical flow through II will be just offset in this case by a downward displacement of the dividing streamline. Thus, if a symmetry condition or a free-slip boundary at infinity is imposed on the free stream, α_1 is determined uniquely and the $O(R^{-\frac{1}{2}})$ correction to the uniform flow in I vanishes identically.

As mentioned earlier, if β becomes sufficiently small, the range of validity of this analysis, in terms of Rr, is somewhat restricted. To determine the extent of this limitation, we note that, in order for the solution to be valid, the thickness of boundary layers II and IV must be much less than that of region III. From (2.1) the thickness of II is of $O(R^{-\frac{1}{2}}r^{\frac{1}{2}})$ while in IV the boundary-layer thickness, as indicated by (2.8), becomes of $O(R^{-\frac{1}{4}}r^{\frac{3}{4}}\beta^{\frac{1}{2}})$ as $\beta \to 0$. Thus, if both boundary layers are to be thin in comparison with region III, whose thickness is of $O(r\beta)$, we must require that $\beta \gg O[(Rr)^{-\frac{1}{2}}]$ for the above analysis to remain valid. It is then evident that, for a given R, the solution as given above applies only beyond a certain distance downstream of the leading edge which increases as β approaches zero.

To summarize then, the absence of a characteristic length in this system has made it possible to obtain similar solutions for the fluid motion throughout the flow field and to derive expressions for the stream functions in regions I–IV, given by (2.10), (2.1), (2.5) and (2.8) respectively. These first-order results describe a solution in which, beneath the free stream, an inviscid separated region forms which, to this order in the analysis, is irrotational and is characterized by a reverse flow of $O[(Rr)^{-\frac{1}{2}}]$ in magnitude, induced by the free shear boundary layer located between I and III. Since the inviscid motion in III cannot satisfy the no-slip requirement at $\theta = \frac{1}{2}\pi\beta$, a second boundary layer, IV, appears along the inclined surface of the wedge and in this an $O[(Rr)^{-\frac{1}{2}}]$ reverse flow accelerates toward the leading edge. In addition, decay of vorticity near the outer edge of IV indicates that the vorticity in III will be of $O(R^{-\frac{3}{4}}r^{-\frac{5}{4}})$. This solution is then valid throughout that region of the flow field where the condition $\beta \gg (Rr)^{-\frac{1}{2}}$ is satisfied.

3. Higher order analysis

Having examined the basic features of the motion, we now seek to extend the validity of the solutions, in terms of Rr, by considering higher order approximations to the flow in the various regions. In this higher order analysis, the solution must continue to remain independent of the choice of the length scale l; thus, in constructing the appropriate expansions we are guided by the fact that each member of the series must retain the self-similar form of the corresponding first-order term. Furthermore, since the first-order expressions for the stream functions in III and IV differ in magnitude by $O[(Rr)^{-\frac{1}{4}}]$, we must initially proceed in powers of $(Rr)^{-\frac{1}{4}}$ in order to satisfy the matching requirements between adjacent regions. Of course, the expansions will become more complicated when eigensolutions begin to appear in the higher order terms, which, in the present case, occurs first in the $O(R^{-1})$ term of the expansion for the free shear boundary-layer solution (Klemp & Acrivos 1972*a*). The appearance of this eigensolution then makes it necessary to add a logarithmic term to the expansion in II which, through the matching, will produce corresponding logarithmic terms in expansion for the other regions. Consequently, the appropriate expressions for the stream function in I–IV become

$$\Psi_{\mathrm{I}}(r,\theta) = r\sin\theta + (r/R)^{\frac{1}{2}}\{f_{11}(\theta) + (Rr)^{-\frac{1}{2}}[f_{12}(\theta)\ln(Rr) + f_{13}(\theta)] + \ldots\}, \tag{3.1}$$

$$\Psi_{\mathrm{II}}(r,\eta) = (r/R)^{\frac{1}{2}}\{f_{21}(\eta) + (Rr)^{-\frac{1}{2}}[f_{22}(\eta)\ln(Rr) + f_{23}(\eta)] + \ldots\}, \tag{3.2}$$

$$\Psi_{\mathrm{III}}(r,\theta) = (r/R)^{\frac{1}{2}}\{f_{31}(\theta) + (Rr)^{-\frac{1}{4}}f_{32}(\theta) + (Rr)^{-\frac{1}{2}}[f_{33}(\theta)\ln(Rr) + f_{34}(\theta)] + \ldots\}, \tag{3.3}$$

$$\Psi_{\mathrm{IV}}(r,\zeta) = (r/R)^{\frac{1}{2}}\{(Rr)^{-\frac{1}{4}}C_1(\beta)f_{41}(\zeta) + (Rr)^{-\frac{1}{2}}C_2(\beta)f_{42}(\zeta) + (Rr)^{-\frac{3}{4}} \times [C_3(\beta)f_{43}(\zeta)\ln(Rr) + C_4(\beta)f_{44}(\zeta)] + \ldots\}, \tag{3.4}$$

in which $f_{11}(\theta), f_{21}(\eta), f_{31}(\theta)$ and $f_{41}(\zeta)$ represent the first-order solutions derived in §2. The functions $C_n(\beta)$, with $C_1(\beta) = [-\frac{1}{2}\gamma_1\operatorname{cosec}\frac{1}{4}\pi\beta]^{\frac{1}{2}}$, have been included in (3.4) so as to render the $f_{4n}(\zeta)$ independent of β. Also, in extending this analysis to higher order, additional corrections to the position of the $\Psi = 0$ streamline must be considered, hence the expansion for $\theta_s(r;R)$ becomes

$$\theta_s(r;R) = (Rr)^{-\frac{1}{2}}\{\alpha_1 + (Rr)^{-\frac{1}{2}}[\alpha_2\ln(Rr) + \alpha_3] + \ldots\}. \tag{3.5}$$

Note that no terms of $O(R^{-\frac{3}{4}})$ appear in (3.1), (3.2) and (3.5) owing to the absence of an $O(R^{-\frac{1}{4}})$ flow in III.

The eigensolution, mentioned above, is represented in (3.2) by $f_{22}(\eta)$, which corresponds to an eigenfunction of the appropriate boundary-layer equation for $f_{23}(\eta)$. It is given (Klemp & Acrivos 1972*a*) by

$$f_{22}(\eta) = K[f'_{21}(\eta)/f'_{21}(0) - 1], \quad f'_{21}(0) = 0{\cdot}5873, \tag{3.6}$$

where K is a constant. In order to determine its effect on the solution and the limitations it imposes, we shall begin the higher order analysis with the boundary-layer region II, where substitution of (3.2) into the radial component of the Navier–Stokes equations and use of (3.6) yields the following ordinary differential equation for $f_{23}(\eta)$

$$2f'''_{23} + f_{21}f''_{23} + f'_{21}f'_{23} = 2Kf''_{21}. \tag{3.7}$$

The boundary conditions for the above as $\eta \to \pm\infty$ derive from the requirement that $f'_{23}(\eta)$ must match, respectively, with the radial component of the $O[(Rr)^{-\frac{1}{2}}]$ velocity in I and III. Thus, evaluating (2.7) as $\theta \to 0$ and recalling that $f'_{11}(0) = 0$, we have that

$$f_{23}(0) = f'_{23}(\infty) = 0, \quad f'_{23}(-\infty) = \tfrac{1}{2}\gamma_1\cot\tfrac{1}{4}\pi\beta. \tag{3.8}$$

By integrating (3.7) once from ∞ to η, we then obtain

$$f''_{23} + \tfrac{1}{2}f_{21}f'_{23} = K[f'_{21} - 1],$$

which clearly indicates that, if $f'_{23}(\eta)$ is to satisfy the required boundary condition at $\eta \to -\infty$, K must be given by

$$K = \tfrac{1}{4}\gamma_1^2\cot\tfrac{1}{4}\pi\beta. \tag{3.9}$$

Notice that K vanishes as $\beta \to -2$. This agrees with the result found by Klemp & Acrivos (1972*b*) that in the boundary layer formed by the mixing of the two parallel streams the first eigensolution does not introduce a logarithmic term in the analysis if one of the streams is semi-infinite.

From the above, we see then that the eigenfunction $f_{22}(\eta)$ is uniquely determined and that it plays an important role in the solution, because, by appearing as a logarithmic term in the expansion (3.2), it introduces a non-homogeneous term in (3.7) which allows $f_{23}(\eta)$ to assume the proper form as $\eta \to -\infty$. However, the fact that $f_{22}(\eta)$ satisfies the homogeneous portion of (3.7) precludes the uniqueness of $f_{23}(\eta)$ because an unknown constant appears in its solution that cannot be determined without examining the details of the flow in the vicinity of the leading edge. This indeterminancy will then be transferred to the expressions for $f_{13}(\theta)$, $f_{34}(\theta)$, $f_{44}(\theta)$ and α_3 through the matching of (3.2) to the stream-function expansions in neighbouring regions. Consequently, in what follows, we shall terminate our analysis after obtaining the logarithmic terms in (3.1)–(3.5).

Turning next to the free-stream region, we can quickly conclude, by matching the pressure across the free shear boundary layer, that the function $f_{12}(\theta)$ in (3.1) must vanish. This is because, as was mentioned in § 2, the pressure in III is of $O[(Rr)^{-1}]$ and thus, since the pressure drop across II is of the same magnitude, any $O[(Rr)^{-1}\ln(Rr)]$ pressure appearing in I must vanish as $\theta \to 0$. This, in turn, requires that $f'_{12}(0) = 0$, and thus the problem for $f_{12}(\theta)$ becomes

$$\nabla^2 f_{12} = (1/r^2)f''_{12}(\theta) = 0, \quad f'_{12}(0) = f_{12}(\pi) = 0,$$

which has only a trivial solution. This result also allows us to calculate α_2 by requiring that the form of (3.2) as $\eta \to \infty$, when expressed in outer variables r and θ, should contain no logarithmic terms to the present order of the analysis. Thus, the logarithmic term in (2.11) arising from (3.5) must cancel with that containing $f_{22}(\infty)$, which, on account of (3.6), yields

$$\alpha_2 = K[1 - 1/f'_{21}(0)] = -0{\cdot}7028K. \tag{3.10}$$

In considering higher order solutions for the inviscid separated region, we are aided by the fact that, as discussed in the previous section, the vorticity in III is of $O(R^{-4}\Psi_{\text{III}}^{-5})$; thus, to the present order of approximation, the flow remains irrotational. With this in mind, we can proceed to construct solutions for $f_{32}(\theta)$ and $f_{33}(\theta)$ using Laplace's equation with the appropriate boundary conditions arrived at through the matching to the adjacent boundary layers. Turning first to the free shear boundary layer, we note that there is no term of $O(R^{-\frac{3}{4}}r^{\frac{1}{4}})$ in (3.2), so that, evaluating (3.6) as $\eta \to -\infty$ and matching (3.2) and (2.3) in the area of overlap, we obtain for the boundary conditions at $\theta = 0$

$$f_{32}(0) = 0, \quad f_{33}(0) = -K.$$

Similarly we see that in IV there is no term of $O[R^{-1}\ln(Rr)]$, so that matching (3.3) as $\theta \to \frac{1}{2}\pi\beta$ to (3.4) as $\zeta \to \infty$, and using (2.10), yields

$$f_{32}(\tfrac{1}{2}\pi\beta) = \lambda_1[-\tfrac{1}{2}\gamma_1 \operatorname{cosec} \tfrac{1}{4}\pi\beta]^{\frac{1}{2}}, \quad f_{33}(\tfrac{1}{2}\pi\beta) = 0, \quad \lambda_1 = 1{\cdot}2585.$$

With these boundary conditions, substitution of (3.3) into Laplace's equation results in the following simple expressions for $f_{32}(\theta)$ and $f_{33}(\theta)$:

$$f_{32}(\theta) = \lambda_1[-\tfrac{1}{2}\gamma_1 \operatorname{cosec}\tfrac{1}{4}\pi\beta]^{\frac{1}{2}} \frac{\sin\frac{1}{4}\theta}{\sin\frac{1}{8}\pi\beta}, \tag{3.11}$$

$$f_{33}(\theta) = K[2\theta/\pi\beta - 1]. \tag{3.12}$$

Since $\lambda_1 > 0$ and $K < 0$, equations (3.11) and (3.12) indicate that the second term in (3.3) corresponds to a reverse flow which is being entrained into IV, while the third term is a result of an $O(R^{-1})$ detrainment from II. As was mentioned earlier, since the expression for $f_{34}(\theta)$ will contain an unknown constant, we shall terminate the analysis for the flow in III at this point.

In order to satisfy the no-slip requirement at $\theta = \frac{1}{2}\pi\beta$ in the higher order analysis, additional corrections to the flow in IV must appear to compensate for the higher order terms derived above for region III. The appropriate equations for these terms in IV are then obtained in the usual manner by substituting (3.4) into the radial component of the momentum equation, thereby yielding

$$\left.\begin{aligned} 4f'''_{42} + f_{41}f''_{42} - 5[1 - f'_{41}f'_{42}] &= 0,\\ 4f'''_{43} + f_{41}f''_{43} - f''_{41}f_{43} - 6[1 - f'_{41}f'_{43}] &= 0.\end{aligned}\right\} \tag{3.13}$$

Also, by suitably choosing $C_2(\beta)$ and $C_3(\beta)$, we obtain for the boundary conditions of (3.13)

$$f_{4n}(0) = f'_{4n}(0) = 0, \quad f'_{4n}(\infty) = -1 \quad (n = 2, 3). \tag{3.14}$$

Thus, $f_{42}(\zeta)$ and $f_{43}(\zeta)$ are clearly independent of β, and can be computed from the integration of (3.13) subject to (3.14). To obtain the β dependence of the terms in (3.4), the radial velocity in IV as $\zeta \to \infty$ must be matched to that in III as $\theta \to \frac{1}{2}\pi\beta$, which, on account of (3.11) and (3.12), leads to

$$C_2(\beta) = -\tfrac{1}{4}\lambda_1 \cot\tfrac{1}{8}\pi\beta, \quad C_3(\beta) = 2K/\pi\beta C_1(\beta). \tag{3.15}$$

Finally, in considering the form of the equations in (3.13) as $\zeta \to \infty$, we find that each similar function has the same asymptotic form as the corresponding first-order function, $f_{41}(\zeta)$, i.e.

$$f_{4n} \xrightarrow[\zeta\to\infty]{} -\zeta + \lambda_n + O(\zeta^{-3}) \quad (n = 1, 2, 3), \tag{3.16}$$

where, from the numerical integration of (3.13), $\lambda_2 = 0{\cdot}1955$ and $\lambda_3 = 0{\cdot}1340$. From (3.16), it is apparent that, as was the case with the first-order solution, both $f''_{42}(\zeta)$ and $f''_{43}(\zeta)$ decay algebraically like ζ^{-5} when ζ becomes large. This confirms that, in fact, the leading term for the vorticity in region III will be of $O(R^{-4}\Psi_{\text{III}}^{-\frac{5}{3}})$ and that higher order approximations to the vorticity in III will then match these higher order terms in the boundary layer IV.

4. Discussion

By taking into account higher order corrections to the basic solution presented in §2, we have thus extended the accuracy of the analysis to $O[R^{-1}\ln(Rr)]$.

However, it is perhaps of more importance that, through the derivation of these higher order terms, we have shown that the proposed solution continues to be self-consistent, in that it represents the first few terms of an apparently well-posed asymptotic expansion. According to this solution, the flow is characterized by a free stream that detaches from the inclined surface at the leading edge and proceeds downstream essentially undisturbed, flowing over a separated region of reverse flow in which the velocity is of $O[(Rr)^{-\frac{1}{2}}]$ in magnitude. Since, in the free-stream region, the singularity has now been removed from the inviscid solution, it appears that this description of the flow structure is more satisfying than that discussed by Hartree (1937) and by Stewartson (1954).

To be sure, as was mentioned previously, the free-slip boundary condition upstream of the leading edge of the inclined surface is somewhat artificial. It has been incorporated into the present analysis, however, in order to maintain an exact analogy between this problem and that described by the Falkner–Skan equation for $\beta < 0$. It is important to note, though, that our present solution can also be applied with relatively minor modifications to situations which are of greater physical interest. Consider first the problem of uniform entry of a stream directly above the leading edge of the wedge surface, i.e. $U(r, \frac{1}{2}\pi) = 1$. In this case, we suppose that the flow in the main stream will separate immediately for the same reasons as those outlined in §§ 1 and 2 and that it will continue downstream undisturbed to a first approximation. Thus, the flow field will again be depicted by figure 2 and the first-order solution in each region will be unchanged. Only higher order terms will be altered owing to vorticity, of $O(R^{-\frac{1}{2}})$ in magnitude, which arises in the main stream because of the uniform entry boundary condition (Van Dyke 1971).

Another interesting case is that in which the horizontal boundary upstream of the leading edge consists of a no-slip surface of finite length. Here, we again suppose that, in spite of the presence of a Blasius boundary layer upstream of the corner, the flow will separate from the inclined surface at the leading edge to avoid a singularity in the potential solution. However, because of the existence now of a physical length, the solution will not have a self-similar form in the vicinity of the corner, where, on account of the boundary layer forming upstream, the free shear boundary layer II will have a finite thickness. Nevertheless, since the free-shear layer is known to approach, asymptotically far downstream, its self-similar structure irrespective of the shape of its profile at $r = 0$, we should also expect our similarity solution, described previously, to apply in all four regions provided that $r \gg 1$.

In all the cases considered above, we have consistently assumed that separation will occur at $r = 0$ which will then lead to the flow pattern given by our solution. Of course, this is not the only possibility. Specifically, if the angle of inclination of the surface changes sufficiently smoothly from $\theta = 0$ to $\theta = \frac{1}{2}\pi\beta$, rather than abruptly as in figures 2 and 3, the adverse pressure gradient may not be strong enough to cause the boundary layer to separate if $-0{\cdot}1988 < \beta < 0$. In this case, the flow pattern far downstream would then be described by the classical analysis based on the Falkner–Skan equation. On the other hand, if the adverse pressure gradient is large enough to cause separation, the angle of inclination of the dividing streamline, θ_s, will again be essentially zero far downstream (as depicted in figure 2) since, for $\theta_s > 0$, infinite velocities would develop far downstream, while, for $\theta_s < 0$, the pressure distribution in the region of inviscid reverse flow would be inconsistent with the required flow pattern. Therefore, the solution for $r \gg 1$ will correspond either to that derived in §§2 and 3 or to that given by the Falkner–Skan analysis depending on whether or not the boundary layer becomes detached from the surface at $r = 0$. Finally, for $\beta < -0{\cdot}1988$, we again note that only the present solution applies far downstream since, as was remarked earlier, the solutions to the Falkner–Skan equation are unacceptable on physical grounds.

Of course, if the separated region is finite in extent, closed streamlines along which the velocity is of order one in magnitude may form in this region. The structure of the separated flow would then be analogous to that which occurs when a uniform stream flows over a finite cavity and would consist of an inviscid core with uniform $O(1)$ vorticity plus an associated boundary layer along the sides. Apparently then, for motions of this type, the semi-infinite solution, as presented above, would not apply except perhaps in the vicinity of the leading edge, where the $O(1)$ velocity of the recirculating fluid vanishes. However, this matter deserves further study since it is not clear to what extent the $O(1)$ vorticity in the core would be convected into the neighbourhood of the leading edge and thereby affect the analysis, as presented earlier, for region III of figure 2.

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